Homological methods in commutative algebra - Summary

D. Excursion: Derived functors, Ext

<u>Definition D.1</u>: A <u>homological δ-functor</u> $F_*: \mathcal{A} \longrightarrow \mathcal{B}$ is a sequence of additive functors $F_i: \mathcal{A} \longrightarrow \mathcal{B}$ with a sequence of natural transformations $\delta_i: F_{i+1}(A'') \longrightarrow F_i(A')$ for ses $0 \to A' \to A \to A'' \to 0$, satisfying the long exact homology sequence

$$\cdots \longrightarrow F_{i+1}(A'') \xrightarrow{\partial} F_i(A') \longrightarrow F_i(A) \longrightarrow F_i(A'') \xrightarrow{\partial} F_{i-1}(A') \longrightarrow \cdots$$

<u>Morphisms of homological δ-functors</u> are sequences $\varphi_i: F_i \longrightarrow G_i$ commuting with the differentials.

A <u>cohomological δ-functor</u> has connecting morphisms $\delta_i: F^i(A'') \to F^{i+1}(A')$ raising the dimension instead.

A <u>left-derived functor</u> of a right exact functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is a homological functor $L_*F: \mathcal{A} \longrightarrow \mathcal{B}$ with $L_0F \cong F$, s.t. for any homological functor $\Phi_*: \mathcal{A} \longrightarrow \mathcal{B}$ and natural transformation $\Phi_0 \longrightarrow L_0F$, there is a unique extension to $\Phi_* \longrightarrow L_*F$.

A <u>right-derived functor</u> of a left exact functor $F: \mathcal{A} \to \mathcal{B}$ is a cohomological functor $R^*F: \mathcal{A} \longrightarrow \mathcal{B}$ with $R^0F \cong F$ and

$$\mathrm{Hom}_{\mathrm{CohomFct}}(R^*F, \Phi^*) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Fct}}(R^0F, \Phi^0)$$

<u>Definition D.2</u>: An object I in an Abelian category is <u>injective</u> if

a) for every mono $\xi: X \longrightarrow Y$, any $\iota: X \longrightarrow I$ extends to some $Y \longrightarrow I$.

$$\begin{tikzcd} & I \\ X \arrow[ur, "\iota"] \arrow[r, "\xi"'] & Y \arrow[u, dashed] \end{tikzcd}$$

b) any ses. $0 \longrightarrow I \longrightarrow X \longrightarrow X'' \longrightarrow 0$ splits.

<u>Definition D.3</u>: An injective resolution of X is a l.es.

$0 \longrightarrow X \xrightarrow{\xi} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$ with all I^i injective.

<u>Proposition D.1</u>: a) Let $\xi: X \longrightarrow I^*$, $\upsilon: Y \longrightarrow J^*$ be injective resolutions, and $f: X \longrightarrow Y$ a morphism. Then there exists a morphism $\varphi: I^* \longrightarrow J^*$ compatible with f (i.e. $\upsilon f = \varphi^0 \xi$.)

If $\tilde{\varphi}$ is a different such morphism, there is a chain homotopy $s^*: I^* \longrightarrow J^{*-1}$ between φ and $\tilde{\varphi}$, i.e. $\varphi^\ell - \tilde{\varphi}^\ell = d_{J^*}^{\ell-1} s^\ell + s^{\ell-1} d_{I^*}^\ell$

(So they induce the same morphism on cohomology.)

b) If $\xi: X \longrightarrow I^*$, $\upsilon: Y \longrightarrow J^*$ are injective resolutions and $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ is a ses., there is an injective resolution $k: Z \longrightarrow K^*$ in a comm. diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\xi"] & Z \arrow[r] \arrow[d, "k"] & Y \arrow[r] \arrow[d, "\upsilon"] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & 0 \\
& {} & {} & {} &
\end{tikzcd}$$

<u>Theorem D.1</u>: Let $\mathcal{A}$ be an Abelian category with sufficiently many injective objects.

a) Any left exact functor $\mathcal{A} \to \mathcal{B}$ has a right derived functor.

b) A cohomological functor $\Phi^*: \mathcal{A} \to \mathcal{B}$ is a right-derived functor of Φ^0 iff $\Phi^i I = 0$ for all injective objects I, $i>0$.

c) Let $0 \to F' \to F \to F'' \to 0$ be a sequence of left exact functors which is exact on injective objects. Then there is a unique sequence of nat. transf. $R^iF'' \to R^{i+1}F'$ s.t. we have for all $X \in Ob(\mathcal{A})$ a l.e.s.

$$0 \to F'X \to FX \to F''X \to R^1F'X \to R^1FX \to R^1F''X \to \dots$$
$$\dots \to R^{i-1}F''X \to R^iF'X \to R^iFX \to R^iF''X \to R^{i+1}F'X \to \dots$$

such that the following square is anti-commutative:

$$\begin{array}{ccc} R^iF'X'' & \xrightarrow{d_F} & R^{i+1}F'X'' \\ \downarrow d_{R^*F''} & & \downarrow d_{R^*F'} \\ R^{i+1}F'X' & \xrightarrow{d_F} & R^{i+2}F'X' \end{array}$$

<u>Theorem D.2</u>: Let $\mathcal{A}$ have sufficiently many projectives.

a) Any right exact functor $F: \mathcal{A} \to \mathcal{B}$ has a left-derived functor.

b) $\Phi_*: \mathcal{A} \to \mathcal{B}$ left-derived functor of Φ_0 iff $\Phi_i P = 0$, P projective

c) For a ses. $0 \to F' \to F \to F'' \to 0$ exact on projectives, there is a l.es.

$$\dots \to L_{p+1}F''X \to L_pF'X \to L_pFX \to L_pF''X \to L_{p-1}F'X \to \dots$$

<u>Definition D.4</u>: If $\mathcal{A}$ has sufficiently many injectives, let for fixed $X \in Ob(\mathcal{A})$ $Ext^*(X,-)$ be the right derived functor of

$$Y \mapsto Hom(X,Y)$$

This is functorial in X. Explicitly, if $Y \to I^*$ is an injective resolution, then

$$Ext^p(X,Y) = H^p(Hom(X, I^*))$$

Since $Ext^p(-,Y)$ vanishes on projectives ($Hom(P,I^*)$ being exact), it is also a right derived functor of $X \mapsto Hom(X,Y)$, and thus if $P^* \to X$ is a projective resolution

$$Ext^p(X,Y) = H^p(Hom(P^*,Y))$$

<u>Proposition D.2</u>: Let $F: \mathcal{A} \to \mathcal{B}$ left exact, $\mathcal{A}$ suff. many injectives. Suppose $\mathcal{X} \subseteq Ob(\mathcal{A})$ satisfies

a) If $X \cong Y \oplus Z$, then $X \in \mathcal{X} \iff Y \in \mathcal{X}$ and $Z \in \mathcal{X}$.

b) Every object in $\mathcal{A}$ has a mono into an object in $\mathcal{X}$.

c) If $0 \to X \to Y \to Z \to 0$ exact and $X, Y \in \mathcal{X}$, then $Z \in \mathcal{X}$ and $0 \to FX \to FY \to FZ \to 0$ is exact.

Then $\mathcal{X}$ contains all injectives and $R^pFX = 0$ when $X \in \mathcal{X}$, $p>0$.

1. Tor and Ext of R-modules

1.1 Injective and projective modules + properties of $\mathrm{Ext}^*_R(-,-)$

Proposition 1.1.1 (Baer's criterion): For an R-module N, t.f.a.e.:
a) $\mathrm{Hom}_R(-, N)$ is an exact functor.
b) If $M \subseteq M'$ is a sub-R-module, every $f: M \longrightarrow N$ extends to an $\varphi: M' \longrightarrow N$
c) For all ideals $I \subseteq R$, every $f: I \longrightarrow N$ extends to $R \longrightarrow N$.

Definition 1.1.0: Such R-module N is called <u>injective</u>

Definition 1.1.1: If R is a domain, an R-module M is called <u>divisible</u> if $M \xrightarrow{\cdot r} M$ is surjective for all $r \in R \setminus \{0\}$.
(Equivalently: c) above holds for $I = (r)$)

Corollary 1.1.1: β) If R is a domain, N injective $\Rightarrow$ N divisible.
γ) If N injective, $S \subseteq R$ multiplicative, then $N \longrightarrow N_S$ surjective.

Corollary 1.1.2: If R is a PID, N injective $\Longleftrightarrow$ N divisible.

Proposition 1.1.2: For any ring R, the category of R-modules has sufficiently many injective objects (and thus injective resolutions.)

Lemma 1.1.1 a) The functor $Ab \longrightarrow R\text{-mod}: G \longmapsto \mathcal{R}G := \mathrm{Hom}_{\mathbb{Z}}(R, G)$ (which is an R-module by $(r\varphi)(\rho) := \varphi(\rho r)$.) is a left adjoint functor to the forgetful functor $R\text{-mod} \longrightarrow Ab$.
b) If I is an injective Abelian group, $\mathcal{R}I$ is an injective R-module.
c) If $M \longrightarrow \mathcal{R}I$ corresponds to an injective morphism $i: M \longrightarrow I$ of Abelian groups under

$$\mathrm{Hom}_{\mathbb{Z}}(M, I) \cong \mathrm{Hom}_R(M, \mathcal{R}I),$$

then $M \longrightarrow \mathcal{R}I$ is also injective.

Corollary (of prop 1.1.2 + cor 1.1.2) 1.1.3: When R is a PID, any quotient of an injective module is injective.
~~Th~~ Thus every R-module has an injective resolution of length 1:

$$0 \longrightarrow X \hookrightarrow I^0 \longrightarrow I^0/X \longrightarrow 0 \longrightarrow \cdots$$

Proposition 1.1.3: a) For a PID R, $\mathrm{Ext}^p_R(M, N) = 0$ for $p > 1$.
b) Any submodule of a projective R-module is projective. (R PID)

Remark: For a PID R, free R-modules are the same as projective R-modules.

Example: For $a \in R$ not a zero-divisor

$$\mathrm{Ext}^k_R(R/aR, M) \cong \begin{cases} \ker(M \xrightarrow{a} M) & k=0 \\ M/aM & k=1 \\ 0 & k \geq 2 \end{cases}$$

Proposition 1.1.4: Let R be any ring, N an R-module. Then N is injective iff for all ideals $I \subseteq R$, $\mathrm{Ext}^1_R(R/I, N) = 0$.

Fact 1.1.1: The multiplication with $a \in R$ on the R-module $\mathrm{Ext}^k_R(M, N)$ is th[e] same as the map induced by either $M \xrightarrow{\cdot a} M$ or $N \xrightarrow{\cdot a} N$.

Fact 1.1.2: If multiplication by a annihitates M or N, then also $\mathrm{Ext}^k_R(M, N$

Fact 1.1.3: If ~~...~~ R is Noetherian, then any f.g. R-module M has a projective resolution $0 \longleftarrow M \longleftarrow P^0 \longleftarrow P^1 \longleftarrow P^2 \longleftarrow \cdots$ where all P^i are f.g. free R-modules.

Fact 1.1.4: If M and N are f.g. over a Noetherian ring R, then all $\mathrm{Ext}^k_R(M,N)$ f.g. [504]

Fact 1.1.5: Let $S \subseteq R$ be multiplicative.

a) If M is a projective R-module, M_S is a projective R_S-module

b) If N is an injective R_S-module, it is an injective R-module.

Fact 1.1.6: For $S \subseteq R$ multiplicative, M an R-module, X an R_S-module, there is a canonical isomorphism

$$\mathrm{Ext}^k_{R_S}(M_S, X) \xrightarrow{\cong} \mathrm{Ext}^k_R(M, X),$$

compatible with l.e.s. and in degree 0 $\mathrm{Hom}_{R_S}(M_S, X) \xrightarrow{\cong} \mathrm{Hom}_R(M, X)$.

Remark: The natural map $\mathrm{Hom}_R(M,N)_S \longrightarrow \mathrm{Hom}_{R_S}(M_S, N_S)$ is in general neither injective nor surjective.

If M is f.g., it is injective, and if M is finitely presented, it is an iso.

Proposition 1.1.5: Let R be Noetherian, M f.g., S multiplicative, X any R-module. Then there is a canonical isomorphism

$$\mathrm{Ext}^k_R(M, X)_S \xrightarrow{\cong} \mathrm{Ext}^k_{R_S}(M_S, X_S)$$

compatible with l.e.s. and in degree 0 $\mathrm{Hom}_R(M,X)_S \xrightarrow{\cong} \mathrm{Hom}_{R_S}(M_S, X_S)$.

Projective and injective dimension

For an object X in an Abelian category $\mathcal{A}$, define

$$\mathrm{idim}_{\mathcal{A}}(X) := \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p(T, X) \neq 0 \text{ for some object } T \text{ of } \mathcal{A}\}$$

$$\mathrm{pdim}_{\mathcal{A}}(X) = \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p(X, T) \neq 0 \text{ for some object } T \text{ of } \mathcal{A}\}$$

When $\mathcal{A}$ has sufficiently many injective objects, $X \neq 0$,

$$\mathrm{idim}_{\mathcal{A}}(X) = \min\{\ell \mid \text{There is an injective resolution of length } \ell\}$$
$$= \min\{\ell \mid \text{Any inj. res. } 0 \to X \to I^* \text{ has truncated inj. res } 0 \to X \to \tau_{\leq \ell} I^*\}$$

When $\mathcal{A}$ has sufficiently many projective objects, $X \neq 0$

$$\mathrm{pdim}_{\mathcal{A}}(X) = \cancel{\sup}\ \min\{\ell \mid \text{There is a projective resolution of length } \ell\}$$
$$= \min\{\ell \mid \text{Any proj. res. } 0 \leftarrow X \leftarrow P_* \text{ has truncated proj res } 0 \leftarrow X \leftarrow \cdots \leftarrow \ker(P_\ell \to P_{\ell-1})\}$$

In fact, it follows from the proof and Baer's criterion that for R-module X

$$\mathrm{idim}_R(X) = \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p_R(R/I, X) \neq 0 \text{ for some ideal } I\}$$
$$= \sup\{p \in \mathbb{N} \mid \mathrm{Ext}^p_R(T, X) \neq 0 \text{ for some f.g. } R\text{-module } T\}$$

~~and similarly for $\mathrm{pdim}_R(X)$~~ not! for pdim! (We don't have proposition 1.1.4)

Corollary 1.1.4: For R Noetherian, $\mathrm{idim}_R(M) = \sup_{\mathfrak{m} \in \mathrm{mSpec}(R)} \mathrm{idim}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \sup_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{idim}_{R_\mathfrak{p}}(M_\mathfrak{p})$

Corollary 1.1.5: For R Noetherian, M f.g.

$$\mathrm{pdim}_R(M) = \sup_{\mathfrak{m} \in \mathrm{mSpec}(R)} \mathrm{pdim}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \sup_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{pdim}_{R_\mathfrak{p}}(M_\mathfrak{p})$$

Corollary 1.1.6: Let R be a Dedekind domain. Then an R-module M is injective iff it is divisible. So any quotient of an injective module is injective. We have $\mathrm{idim}_R(M) \leq 1$ for all M.

Also $\mathrm{pdim}_R(M) \leq 1$ for all M. Any submodule of a projective R-module is projective. Any f.g. torsion free R-module is projective.

Proposition 1.1.6: Let R be Noetherian, $I \subseteq R$ ideal, M, N f.g. R-modules. Then

$$\widehat{\mathrm{Ext}^p_R(M,N)} = \mathrm{Ext}^p_{\hat{R}}(\hat{M}, \hat{N}) \qquad (I\text{-adic completion})$$

Lemma 1.1.2: An R-module P is projective iff it is a direct sum of a free R-module. If P is f.g., F may be chosen f.g. as well.

Corollary 1.1.8: If P is finitely represented, then P is projective iff $\mathrm{Ext}^1_R(P,T)=0$ for all f.g. R-modules T.

Corollary 1.1.7: If R Noetherian, M f.g., then

$$\mathrm{pdim}_R(M) \geq \mathrm{pdim}_{\hat{R}}(\hat{M})$$

When R is local and $I \subseteq R$ is a proper ideal, equality holds.

1.2 Torsion products and flat modules

Definition 1.2.1: For an R-module M, let $\mathrm{Tor}^R_p(M,N)$ be the p-th left derived functor of $N \longmapsto M \otimes_R N$.

Remark: For a s.e.s. $0 \to N' \to N \to N'' \to 0$, we get a l.e.s.

$$\to \mathrm{Tor}^R_{p+1}(M,N'') \xrightarrow{\partial} \mathrm{Tor}^R_p(M,N') \to \mathrm{Tor}^R_p(M,N) \to \mathrm{Tor}^R_p(M,N'') \xrightarrow{\partial} \mathrm{Tor}^R_{p-1}(M,N') \to$$

For a s.e.s. $0 \to M' \to M \to M'' \to 0$, we get the sequence

$$0 \to M' \otimes_R - \to M \otimes_R - \to M'' \otimes_R - \to 0$$

which is exact on free, and hence on projective, R-modules, giving a l.e.s.

$$\to \mathrm{Tor}^R_{p+1}(M'',N) \xrightarrow{\partial} \mathrm{Tor}^R_p(M',N) \to \mathrm{Tor}^R_p(M,N) \to \mathrm{Tor}^R_p(M'',N) \xrightarrow{\partial} \mathrm{Tor}^R_{p-1}(M',N) \to \cdots$$

To calculate $\mathrm{Tor}^R_i(M,N)$, one takes a projective resolution $P_* \to N \to 0$ and then $\mathrm{Tor}^R_i(M,N) = H_i(M \otimes P_*)$.

Fact 1.2.1: If $i>0$ and one of M,N is projective, then $\mathrm{Tor}^R_i(M,N)=0$.

So $\mathrm{Tor}^R_i(-,N)$ is also the left-derived functor of $M \longmapsto M \otimes N$.

Proposition 1.2.1: There is a unique family of isomorphisms $\mathrm{Tor}^R_i(M,N) \cong \mathrm{Tor}^R_i(N,M)$ which is $M \otimes N \xrightarrow{\cong} N \otimes M$ in degree 0 and that exchanges the two connecting homomorphisms in the l.e.s.'s above.

Example 1.2.1: For $a \in R$ not a zero-divisor

$$\mathrm{Tor}^R_i(M, R/aR) \cong \begin{cases} M/aM & i=0 \\ \ker(M \xrightarrow{a} M) & i=1 \\ 0 & \text{otherwise} \end{cases}$$

Fact 1.2.2: Multiplication with $a \in R$ on ~~Ext~~ $\mathrm{Tor}^R_i(M,N)$ is the same as the maps induced by $M \xrightarrow{\cdot a} M$ and $N \xrightarrow{\cdot a} N$.

Fact 1.2.3: If R Noetherian, M,N f.g. then $\mathrm{Tor}^R_i(M,N)$ f.g.

Fact 1.2.4: For a multiplicative subset $S \subseteq R$, there are canonical isos $\mathrm{Tor}^R_i(M,N)_S \cong \mathrm{Tor}^R_i(M,N_S) \cong \mathrm{Tor}^R_i(M_S,N) \cong \mathrm{Tor}^R_i(M_S,N_S) \cong \mathrm{Tor}^{R_S}_i(M_S,N_S)$.

Fact 1.2.5: Tor is compatible with direct limits: $\mathrm{Tor}^R_i(M, \varinjlim N_\lambda) = \varinjlim \mathrm{Tor}^R_i(M,N_\lambda)$

Proposition 1.2.2.: For an R-module M, t.f.a.e.:

a) The functor $M \otimes -$ is exact

b) $\mathrm{Tor}^R_i(M,N)=0$ when $i>0$ and any N — $\tilde{b}$) Only for $i=1$

c) $\mathrm{Tor}^R_i(M,N)=0$ when $i>0$ and N f.g. — $\tilde{c}$) Only for $i=1$.

Definition 1.2.2. Such R-module M is called <u>flat</u>.

Proposition 1.2.3: For an R-module M, t.f.a.e.: $\boxed{\text{If } R \text{ Noetherian, b) and c) only necessary for } I \text{ prime}}$

a) M is flat

b) For any ideal $I \subseteq R$, $\mathrm{Tor}^R_1(M, R/I)=0$.

c) For any I, $I \otimes_R M \to IM : i \otimes m \mapsto im$ is injective (hence iso)

Corollary 1.2.1: If M is a flat module, it is torsion-free. The converse holds if R is a Dedekind domain.

Example 1.2.2: If M is a projective module, it is flat.

Fact 1.2.6: For an R-module M, t.f.a.e.

(a) M is flat

(b) For any multiplicative $S \subseteq R$, M_S is a flat R-module (c) —"— R_S-module

(d) For every maximal ideal $\mathfrak{m}$, $M_{\mathfrak{m}}$ is a flat R-module (e) —"— $R_{\mathfrak{m}}$-module

Example 1.2.4: Any localization is a flat R-module (i.e. R_S is flat)

Example 1.2.5: The completion $\hat{R}$ w.r.t. any ideal I is a flat R-module, if R is Noetherian. If M is f.g. and flat, $\hat{M}$ is a flat $\hat{R}$-module.

Example 1.2.6: The product of arbitrarily many flat modules over a Noetherian ring is again flat.

Example 1.2.7. The direct sum of arbitrarily many flat modules is always flat.

Fact 1.2.7: For any R-module M, t.f.a.e.

(a) $\mathrm{Tor}_p^R(M,T) = 0$ for any T and any $p > d$ $\qquad$ $\hat{a}$) Only $p = d+1$

b) $\mathrm{Tor}_p^R(M,T) = 0$ for any f.g. T and any $p > d$ $\qquad$ $\hat{b}$) Only $p = d+1$

c) $\mathrm{Tor}_p^R(M,R/I) = 0$ for any ideal $I \subseteq R$, any $p > d$ $\qquad$ $\hat{c}$) Only $p = d+1$.

For R Noetherian, it is enough to have c or $\hat{c}$) for I prime.

Definition 1.2.3: Define the flat dimension $\mathrm{fl.dim}_R(M)$ of M to be the largest such d (or $+\infty$ if none exists)

Fact 1.2.8: For an R-module M

a) $\mathrm{fl.dim}_R(M) \leq \mathrm{pdim}_R(M)$

b) $\mathrm{fl.dim}_R(M) = \sup_{\mathfrak{m} \in \mathrm{mSpec}(R)} \mathrm{fl.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \sup_{\mathfrak{m}} \mathrm{fl.dim}_R(M_{\mathfrak{m}})$

c) $\mathrm{fl.dim}_R(M) = \sup_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{fl.dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \sup_{\mathfrak{p}} \mathrm{fl.dim}_R(M_{\mathfrak{p}})$

Remark: Examples 1.2.5, 1.2.6, 1.2.7. remain true for $\{R\text{-modules of } \mathrm{fl.dim}_R \leq d\}$

Proposition 1.2.4: For an R-module M, t.f.a.e.

(a) $\mathrm{fl.dim}(M) \leq d$

(b) M has a flat resolution

(4) $\quad 0 \leftarrow M \leftarrow F_0 \leftarrow \cdots \leftarrow F_d \leftarrow 0$

of length d, i.e. all F_i are flat.

(c) For any sequence (4), if $F_0, F_1, \dots, F_{d-1}$ are flat, so is F_d.

If R is Noetherian, and M f.g., all F_i can be taken to be f.g.

Fact 1.2.9: If $0 \to N \to F \to M \to 0$ is a ses. of R-modules and F is flat, then $\mathrm{fl.dim}_R(N) = \max\{0, \mathrm{fl.dim}_R(M) - 1\}$

Proposition 1.2.5: Let R be Noetherian, M, N f.g. Then

$$\mathrm{Tor}_i^{\hat{R}}(\hat{M}, \hat{N}) \cong \widehat{\mathrm{Tor}_i^R(M,N)}$$

1.3 The case of f.g. modules over Noetherian rings

Proposition 1.3.1: Let R be Noetherian local, M f.g. Then t.f.a.e.

(a) M is free
(b) M is projective
(c) M is flat
(d) $\mathrm{Tor}_1^R(M, k) = 0$
(e) $\mathrm{Tor}_1^{\hat{R}}(\hat{M}, k) = 0$ (completion w.r.t. m.)

(For (d) $\Rightarrow$ (a), choose $R^n \xrightarrow{\pi} M$ giving iso when tensored with k and note that $\ker(\pi) \otimes k = 0$ implies $\ker(\pi) = 0$ by Nakayama's lemma.

Corollary 1.3.1: If R Noetherian local, M f.g. then
$$\mathrm{pdim}_R(M) = \mathrm{fl.dim}_R(M) = \sup\{d \in \mathbb{N} \mid \mathrm{Tor}_{d+1}^R(M,k) \neq 0\}$$

Corollary 1.3.2: If R Noetherian local, M <u>any</u> R-module, then
$$\mathrm{fl.dim}_R(M) \leq \mathrm{fl.dim}_R(k).$$

Corollary 1.3.3: If R Noetherian (local or not) and M f.g. then
$$\mathrm{fl.dim}_R(M) = \mathrm{pr.dim}_R(M).$$

Proposition 1.3.2: If R Noetherian, M f.g., t.f.a.e.

(a) M is projective
(b) M is flat
(c) $\mathrm{Spec}\, R$ can be covered by $\mathrm{Spec}\, R_f$, $f \in R$, s.t. M_f is a free R_f-module
(d) $\tilde{M}$ is a locally free $\mathcal{O}_{\mathrm{Spec}\, R}$-module on $\mathrm{Spec}\, R$
(e) M_m is a free R_m-module for any maximal ideal m of R.
(f) M_p is a free R_p-module for any prime ideal p of R.

Non-trivial part is (f) $\Rightarrow$ (c), for which we have:

Lemma 1.3.1: Let R be Noetherian, M f.g. Let $p \in \mathrm{Spec}\, R$ s.t. M_p is a free R_p-module. Then there is $f \in R \setminus p$ s.t. M_f is free R_f-module.

Proof: Let $m_1, \dots, m_k \in M$ s.t. their images in M_p generate this free R_p-module. The finitely many generators of M over R can in M_p be written as a linear combinattion of the m_j, and by finding a common denominator of the coefficients this holds even in R_f for some $f \in R \setminus p$.

Let $N \subseteq R_f^k$ be the kernel of $R_f^k \twoheadrightarrow M_f : (r_i)_{i=1}^k \mapsto \sum_{i=1}^k r_i m_i$.

Then $0 \to N \to R_f^k \twoheadrightarrow M_f \to 0$ gives $0 \to N_p \to R_p^k \xrightarrow{\cong} M_p \to 0$, implying that $N_p = 0$. As R is Noetherian, N is f.g. Since all finitely many generators get killed by multiplying by some element in $(R_f) \setminus p$, there is $\hat{f} \in (R_f) \setminus p$ s.t. $\hat{f} N = 0$, so $N_{\hat{f}} = 0$ and $(R_{\hat{f}})^k \xrightarrow{\cong} M_{\hat{f}}$, so $M_{\hat{f}}$ is free.

2 Regular rings and Cohen-Macaulay-rings

2.1 An application of the Koszul complex

Let M be an R-module and $\underline{x} = (x_1, \dots, x_n) \in R^n$. We define now the Koszul complex $C_*(\underline{x}, M)$. For $I = \{1, \dots, n\}$, define

- $C_p(\underline{x}, M) = \{\text{Alternating functions } I^{n-p} \xrightarrow{f} M\}$
- $d_j : C_p(\underline{x}, M) \longrightarrow C_{p-1}(\underline{x}, M)$ $\qquad (k := n-p)$
  $$(d_j f)(i_1, \dots, i_{k+1}) = x_{i_{j+1}} f(i_1, \dots, \hat{i}_{j+1}, \dots, i_{k+1})$$
- $d : C_p(\underline{x}, M) \longrightarrow C_{p-1}(\underline{x}, M) \quad : \quad d = \sum_{j=0}^{k} (-1)^j d_j$

For a morphism $f : A_* \longrightarrow B_*$ of chain complexes, define the cone
$$C(f)_i := \mathrm{Cone}(f)_i := B_i \oplus A_{i-1}, \quad d_{C_*}(b, a) = (d_B b + f(a), -d_A(a))$$
The ses. $0 \longrightarrow B_* \longrightarrow C(f)_* \longrightarrow A_*[-1] \longrightarrow 0$ gives the les.
$$\dots \longrightarrow H_i(A_*) \xrightarrow{f_*} H_i(B_*) \longrightarrow H_i(C(f)) \longrightarrow H_{i-1}(A_*) \xrightarrow{f_*} \dots$$

We have for $\underline{x}' = (x_1, \dots, x_{n-1})$
$$C_*(\underline{x}, M) \xrightarrow{\cong} \mathrm{Cone}(C_*(\underline{x}', M) \xrightarrow{\cdot x_n} C_*(\underline{x}, M))$$
$$C_p(\underline{x}, M) \xrightarrow{\cong} C_p(\underline{x}', M) \oplus C_{p-1}(\underline{x}', M)$$
$$f \longmapsto (\cancel{g, h})(f(n, -)_{I_{n-1}^{n-p-1}}, f|_{I_{n-1}^{n-p}})$$

<u>Definition 2.1.1</u> A sequence $(x_1, \dots, x_n)$ is called <u>M-regular</u> if
$$M/(x_1M + \dots + x_{i-1}M) \xrightarrow{\cdot x_i} M/(x_1M + \dots + x_{i-1}M)$$
is injective for $1 \le i \le n$.

<u>Proposition 2.1.1</u> a) If $\underline{x}$ is M-regular, then $H_i(\underline{x}, M) = 0$ for $i > 0$.
b) If R is Noetherian local and M f.g., the converse also holds, if $x_i \in \mathfrak{m}$.
c) $H_0(\underline{x}, M) = M/(x_0M + \dots + x_nM)$.

<u>Corollary 2.1.1</u>: When R is a Noetherian local ring, M f.g., then any permutation of an M regular sequence of elements of $\mathfrak{m}$ stays M-regular.

<u>Fact 2.1.1</u>: Let R be a ring, $\underline{x} = (x_1, \dots, x_n)$ an R-regular sequence such that $I = \underline{x}R$ is a proper ideal. Then

a) $0 \longleftarrow R/I \longleftarrow C_*(\underline{x}, R)$ is a free resolution of R/I.

b) For every R-module M
$$\mathrm{Tor}_i^R(M, R/I) \cong H_i(\underline{x}, M)$$
$$\mathrm{Ext}_R^i(R/I, M) \cong H_{n-i}(\underline{x}, M)$$

c) $\mathrm{pdim}(R/I) = \mathrm{fl.dim}(R/I) = n$.

2.2 Regular rings

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. We have $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \ge d$. We call R <u>regular</u> if equality holds. This is equivalent to $\mathfrak{m}$ being generated by d elements $x_1, \dots, x_d$ and to
$$\mathrm{Gr.}(R, \mathfrak{m}) := \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$
being a polynomial ring. It follows that regular rings are domains.

Proposition 2.2.1: Let R be regular Noetherian local $x_1, \dots, x_d$ generating $\mathfrak{m}$.

a) The sequence $\underline{x} = (x_1, \dots, x_d)$ is a regular sequence.

b) The Koszul complex is a free resolution $0 \leftarrow k = R/\mathfrak{m} \leftarrow C_*(\underline{x}, R)$ of length d.

c) The cohomological dimension of the category of R-modules is d (i.e. the supremum of projective and injective dimensions.)

(Recall prime avoidence lemma)

Theorem 1 (Serre): For a Noetherian local ring t.f.a.e.:

a) R is regular

b) $\mathrm{gl.dim}(R)$ is finite

c) $\mathrm{gl.dim}(R) = \dim(R)$

d) $d := \mathrm{fl.dim}_R(k)$ is finite.

Proof: d) $\Rightarrow$ a). We induction on $d := \mathrm{fl.dim}_R(k)$. If $d = 0$ k is flat over R so by proposition 1.3.1 k is free so $k = R$ and R is regular of dim 0. Now let $d > 0$. Then $\mathfrak{m} \neq \mathfrak{m}^2$ as otherwise $\mathfrak{m} = 0$, $R = k$ and $d = 0$.

Claim: $\mathfrak{m} \notin \mathrm{Ass}_R(R)$. Indeed if $\mathfrak{m} = \mathrm{Ann}_R(r) = \ker(R \xrightarrow{\cdot r} R)$ we have a s.e.s. $0 \longrightarrow k = R/\mathfrak{m} \xrightarrow{\cdot r} R \longrightarrow Q \longrightarrow 0$ giving

$$0 \longrightarrow \mathrm{Tor}^R_{p+1}(T, Q) \xrightarrow{\cong} \mathrm{Tor}^R_p(T, k) \longrightarrow 0$$

so $\mathrm{fl.dim}(k) = \mathrm{fl.dim}(Q) - 1 < \mathrm{fl.dim}(Q) \leq \mathrm{fl.dim}(k)$. $\lightning$

So by prime avoidence there is $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(R)} \mathfrak{p})$. It follows that $R \xrightarrow{\cdot x} R$ is injective so

$$\underline{\mathrm{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m}) \leq \mathrm{pdim}_R(\mathfrak{m}) \overset{\text{cor. 1.3.3}}{=} \mathrm{fl.dim}_R(\mathfrak{m})}.$$

From $0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0$ we get $\mathrm{Tor}^R_k(\mathfrak{m}, T) \cong \mathrm{Tor}^R_{k+1}(k, T)$ for all T, so $\underline{\mathrm{fl.dim}_R(\mathfrak{m}) \leq \mathrm{fl.dim}_R(k) - 1 = d - 1}$.

Let $x = x_1, x_2, \dots, x_e$ be elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ generate this as a k-vector space. This gives a splitting of $0 \longrightarrow k \xrightarrow{x} \mathfrak{m}/x\mathfrak{m} \longrightarrow \mathfrak{m}/xR \longrightarrow 0$, showing that k is a direct summand of the R/xR-module $\mathfrak{m}/x\mathfrak{m}$ and thus

$$\delta := \mathrm{fl.dim}_{R/xR}(k) = \mathrm{pdim}_{R/xR}(k) \leq \mathrm{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m})$$
$$\leq \mathrm{fl.dim}_R(\mathfrak{m}) < d.$$

By the induction hypothesis R/xR is a regular ring of dim $\delta < d$. Let $\xi_1, \dots, \xi_\delta \in \mathfrak{m}$ whose images in R/xR generate the maximal ideal $\mathfrak{m}/xR$. Then $\xi_1, \dots, \xi_\delta, x$ generate $\mathfrak{m}$ so $\dim R \leq \delta + 1$.

It remains to prove that $\delta = \dim(R/xR) < \dim(R)$. Let $\mathfrak{q}_0 \subsetneq \dots \subsetneq \mathfrak{q}_\delta$ be a sequence of prime ideals in R/xR with preimages $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_\delta$ in R. Then $x \in \mathfrak{p}_0$, so $\mathfrak{p}_0 \notin \mathrm{Ass}_R(R)$, so $\mathfrak{p}_0$ is not minimal and thus $\dim(R) > \delta$. So $\dim(R) = \delta + 1$ and R is regular.

Corollary 2.2.1 (Serre): Let R be a regular local ring and $\mathfrak{p} \in \mathrm{Spec}(R)$. Then $R_{\mathfrak{p}}$ is also regular.

Definition 2.2.1: A Noetherian ring R is called regular if
a) For any $\mathfrak{p} \in \mathrm{Spec}\, R$, $R_{\mathfrak{p}}$ is a regular local ring.
b) For any $\mathfrak{m} \in \text{m-Spec}\, R$, $R_{\mathfrak{m}}$ is a regular local ring.

An ideal $I \subseteq R$, then R is called regular in I if
a) For any $\mathfrak{p} \in \mathrm{Spec}\, R$ containing I, $R_{\mathfrak{p}}$ is a regular local ring
b) For any $\mathfrak{m} \in \text{m-Spec}\, R$ containing I, $R_{\mathfrak{m}}$ is a regular local ring.

Proposition 2.2.2: Let R be a Noetherian ring regular in an ideal I. Then the completion $\hat{R}$ of R w.r.t. I is a regular ring.

Proof: For $x \in I \cdot \hat{R}$, $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ converges in $\hat{R}$, so I is contained in the Jacobson radical and thus every maximal ideal $\mathfrak{n}$ of $\hat{R}$ contains I. Its preimage $\mathfrak{m}$ in R is also maximal and contains I, so $R_{\mathfrak{m}}$ is a regular local ring. By $Gr_\cdot(\hat{R}, \mathfrak{n}) \cong Gr_\cdot(R, \mathfrak{m})$, $R_{\mathfrak{m}}$ and $\hat{R}_{\mathfrak{n}}$ have the same Krull dimension, and since generators of $\mathfrak{m}$ give generators of $\mathfrak{n}$, we see that $\hat{R}_{\mathfrak{n}}$ is regular.

Proposition 2.2.3: If R is a regular Noetherian ring, then $R[T_1, \dots, T_n]$ and $R[[T_1, \dots, T_n]]$ are also regular.

Proof: Enough for $R[T]$. Let $\mathfrak{p} \in \mathrm{Spec}\, R[T]$ and $\mathfrak{m} = \mathfrak{p} \cap R$. W.l.o.g. $\mathfrak{m}$ is maximal, and R is local. Let $x_1, \dots, x_d$ be generators of $\mathfrak{m}$.
If $\mathfrak{p} = \mathfrak{m}[T]$, they also generate $\mathfrak{p}$ over $R[T]$, which by $\dim(R[T]_{\mathfrak{p}}) = ht(\mathfrak{p}) \geq ht(\mathfrak{m}) = d$ shows that $R[T]_{\mathfrak{p}}$ is regular.
If $\mathfrak{p} \supsetneq \mathfrak{m}[T]$, then $ht(\mathfrak{p}) > ht(\mathfrak{m}) = d$. W.l.o.g. $\mathfrak{p}$ is maximal, so $k(\mathfrak{p}) = R[T]/\mathfrak{p}$ is a field extension of $k = R/\mathfrak{m}$ of finite type, so a finite extension. If Q is a minimum polynomial of $T \in R[T]/\mathfrak{p}$ then $x_1, \dots, x_d, Q$ generate $\mathfrak{p}$ and $R[T]_{\mathfrak{p}}$ is regular.

Proposition 2.2.4: All f.g. modules over regular local rings have finite free resolutions.

2.3 Regular sequences and depth

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring

Proposition 2.3.1: Let M be a f.g. R-module. For $n \geq 0$, t.f.a.e:
a) There is a f.g. R-module T with $\mathrm{supp}(T) = \{\mathfrak{m}\}$ s.t. $\mathrm{Ext}^j_R(T, M) = 0$ for $j < n$.
b) $\mathrm{Ext}^j_R(k, M) = 0$ for $j < n$
c) For every f.g. R-module T with $\mathrm{supp}(T) = \{\mathfrak{m}\}$, $\mathrm{Ext}^j_R(T, M) = 0$ for $j < n$.

Definition 2.3.1: The largest such n is called the depth of M.

Fact 2.3.1: $\mathrm{Depth}(A \oplus B) = \min(\mathrm{depth}(A), \mathrm{depth}(B))$

Remark 2.3.1: $\mathrm{Supp}\{T\} = \{\mathfrak{m}\}$ is equivalent to $\dim(T) = 0$.

Theorem 2 (Auslander, Buchsbaum) Let R Noetherian local, $M \neq 0$ f.g. $\mathrm{pdim}(M) < \infty$. Then
(i) $\mathrm{pdim}(M) + \mathrm{depth}(M) = \mathrm{depth}(R)$.

Proof: Do induction on $d := \mathrm{pdim}(M)$. If $d=0$, M is free, so $\mathrm{depth}(M) = \mathrm{depth}(R)$.
For $d>0$, let $m_1, \ldots, m_k \in M$ be generators forming a basis of $M/\mathfrak{m}M$. With $P = R^k$ we get a s.e.s. $0 \to M' \xrightarrow{i} P \to M \to 0$ where $P \to M$ sends the i-th basis vector to m_i. By long exact sequence

$$\mathrm{pdim}(M') = \mathrm{pdim}(M) - 1$$

and it suffices to show that $\mathrm{depth}(M) = \mathrm{depth}(M') - 1$.
If $d=1$, M' is free, say $M' = R^l$, and as $i(M') = \ker(P \to M) = \mathfrak{m}P$, i is given by a $(k \times l)$-matrix with coefficients in $\mathfrak{m}$ so $\mathrm{Ext}^j(k, M') \xrightarrow{i} \mathrm{Ext}^j(k, P)$ vanishes for all j. We get s.e.s.

$$0 \to \mathrm{Ext}^j(k, P) \to \mathrm{Ext}^j(k, M) \to \mathrm{Ext}^{j+1}(k, M') \to 0$$

implying that $\mathrm{depth}(M) = \min(\mathrm{depth}(P), \mathrm{depth}(M') - 1)$, which is just $\mathrm{depth}(M') - 1$ by freeness of P and M'.
If $d > 1$, write $c = \mathrm{depth}(M')$. By the IH, $\mathrm{depth}(R) = (d-1) + c \geq c+1$, so for $i \leq c$

$$\underset{\substack{\| \\ 0}}{\mathrm{Ext}^i(k, P)} \to \mathrm{Ext}^i(k, M) \xrightarrow{\cong} \mathrm{Ext}^{i+1}(k, M') \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^{i+1}(k, P)}$$

so $\mathrm{depth}(M) = \mathrm{depth}(M') - 1$ as desired.

Lemma 2.3.2: Let $x \in \mathfrak{m}$ be M-regular. Then

$$\mathrm{depth}(M) = \mathrm{depth}(M/xM) + 1$$

Corollary 2.3.1: If $(x_1, \ldots, x_k)$ is an M-regular sequence of elements of $\mathfrak{m}$, then

$$\mathrm{depth}(M) = \mathrm{depth}\left(M/(x_1M + \ldots + x_kM) \right) + k$$

Proposition 2.3.2: For a f.g. R-module M, t.f.a.e.:

a) $\mathrm{depth}(M) \geq n$

b) Each M-regular sequence $(x_1, \ldots, x_j) \in \mathfrak{m}^j$ for $j \leq n$ can be extended to an M-regular sequence $(x_1, \ldots, x_n) \in \mathfrak{m}^n$.

c) There is an M-regular sequence $(x_1, \ldots, x_n) \in \mathfrak{m}^n$.

Proposition 2.3.3 We have $\mathrm{depth}_{\hat{R}}(\hat{M}) = \mathrm{depth}_R(M)$ (completion w.r.t. $I \subseteq \mathfrak{m}$).

Proposition 2.3.4: If $i < \mathrm{depth}(M) - \dim(T)$, then $\mathrm{Ext}^i(T, M) = 0$.

Proof: Do induction on $d := \dim(T)$, the case $d=0$ being true by definition.